

Signed group orthogonal designs and their applications

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Abstract Craigen introduced and studied *signed group Hadamard matrices* extensively in [1, 2]. Livinskyi [13], following Craigen's lead, studied and provided a better estimate for the asymptotic existence of signed group Hadamard matrices and consequently improved the asymptotic existence of Hadamard matrices. In this paper, we introduce and study signed group orthogonal designs. The main results include a method for finding signed group orthogonal designs for any k -tuple of positive integer and then an application to obtain orthogonal designs from signed group orthogonal designs, namely, for any k -tuple (u_1, u_2, \dots, u_k) of positive integers, we show that there is an integer $N = N(u_1, u_2, \dots, u_k)$ such that for each $n \geq N$, a full orthogonal design (no zero entries) of type $(2^n u_1, 2^n u_2, \dots, 2^n u_k)$ exists. This is an alternative approach to the results obtained in [8].

Keywords Asymptotic existence, Circulant matrix, Hadamard matrix, Orthogonal design, Signed group.

1 Introduction

A *signed group* S (see [2]) is a group with a distinguished central element, an element that commutes with all elements of the group, of order two. Denote the unit of a group as 1 and the distinguished central element of order two as -1. In every signed group, the set $\{1, -1\}$ is a normal subgroup, and we call the number of elements in the quotient group $S/\langle -1 \rangle$ the order of signed group S . So, a signed group of order n is a group of order $2n$. A signed group T is called a *signed subgroup* of a signed

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group S , if T is a subgroup of S and the distinguished central elements of S and T coincide. We denote this relation by $T \leq S$.

Example 1. There are a number of signed groups with different applications. We present some of them used in this work:

- (i) The trivial signed group $S_{\mathbb{R}} = \{1, -1\}$ is a signed group of order 1.
- (ii) The complex signed group $S_{\mathbb{C}} = \langle i; i^2 = -1 \rangle = \{\pm 1, \pm i\}$ is a signed group of order 2.
- (iii) The Quaternion signed group $S_Q = \langle j, k; j^2 = k^2 = -1, jk = -kj \rangle = \{\pm 1, \pm j, \pm k, \pm jk\}$ is a signed group of order 4.
- (iv) The set of all monomial $\{0, \pm 1\}$ -matrices of order n , SP_n , forms a group of order $2^n n!$ and a signed group of order $2^{n-1} n!$.

Let S and T be two signed groups. A *signed group homomorphism* $\phi : S \rightarrow T$ is a map such that for all $a, b \in S$, $\phi(ab) = \phi(a)\phi(b)$ and $\phi(-1) = -1$. A *remrep* (real monomial representation) is a signed group homomorphism $\pi : S \rightarrow SP_n$. A *faithful remrep* is a one to one remrep.

Let R be a ring with unit 1_R , and let S be a signed group with distinguished central element -1_S . Then $R[S] := \{\sum_{i=1}^n r_i s_i; r_i \in R, s_i \in P\}$ is the signed group ring, where P is a set of coset representatives of S modulus $\langle -1_S \rangle$ and for $r \in R$, $s \in P$, we make the identification $-rs = r(-s)$. Addition is defined termwise, and multiplication is defined by linear extension. For instance, $r_1 s_1 (r_2 s_2 + r_3 s_3) = r_1 r_2 s_1 s_2 + r_1 r_3 s_1 s_3$, where $r_i \in R$ and $s_i \in P$, $i \in \{1, 2, 3\}$.

In this work, we choose $R = \mathbb{R}$. Suppose $x \in \mathbb{R}[S]$. Then $x = \sum_{i=1}^n r_i s_i$, where $r_i \in \mathbb{R}$, $s_i \in P$. The *conjugation* of x , denoted \bar{x} , is defined as $\bar{x} := \sum_{i=1}^n r_i s_i^{-1}$. Clearly, the conjugation is an involution, i.e., $\bar{\bar{x}} = x$ for all $x \in \mathbb{R}[S]$, and $\overline{xy} = \bar{y}\bar{x}$ for all $x, y \in \mathbb{R}[S]$. As an example, $\overline{\sqrt{2}j + 3jk} = \sqrt{2}j^{-1} + 3(jk)^{-1} = -\sqrt{2}j - 3jk$, where $j, k \in S_Q$.

For an $m \times n$ matrix $A = [a_{ij}]$ with entries in $\mathbb{R}[S]$ define its adjoint as an $n \times m$ matrix $A^* = \bar{A}^t = [\bar{a}_{ji}]$. Let S be a signed group, and let $A = [a_{ij}]$ be a square matrix such that $a_{ij} \in \{0, \varepsilon_1 x_1, \dots, \varepsilon_k x_k\}$, where $\varepsilon_\ell \in S$ and x_ℓ is a variable, $1 \leq \ell \leq k$. For each $a_{ij} = \varepsilon_\ell x_\ell$ or 0, let $\bar{a}_{ij} = \bar{\varepsilon}_\ell x_\ell$ or 0, and $|a_{ij}| = |\varepsilon_\ell x_\ell| = x_\ell$ or 0. We define $\text{abs}(A) := [|a_{ij}|]$. We call A *quasisymmetric*, if $\text{abs}(A) = \text{abs}(A^*)$, where $A^* = [\bar{a}_{ji}]$. Also, A is called *normal* if $AA^* = A^*A$. The *support* of A (see [2]) is defined by $\text{supp}(A) := \{\text{positions of all nonzero entries of } A\}$.

Suppose $A = (a_1, a_2, \dots, a_n)$ and $B = (b_1, b_2, \dots, b_n)$ are two sequences with elements from $\{0, \varepsilon_1 x_1, \dots, \varepsilon_k x_k\}$, where the x_k 's are variables and $\varepsilon_k \in S$ ($1 \leq k \leq n$) for some signed group S . We use $A_{\bar{R}}$ to denote the sequence whose elements are those of A , conjugated and in reverse order (see [3]), i.e., $A_{\bar{R}} = (\bar{a}_n, \dots, \bar{a}_2, \bar{a}_1)$. We say A is *quasireverse* to B if $\text{abs}(A_{\bar{R}}) = \text{abs}(B)$.

A circulant matrix $C = \text{circ}(a_1, a_2, \dots, a_n)$ (see [6, chap. 4]) can be written as $C = a_1 I_n + \sum_{k=1}^{n-1} a_{k+1} U^k$, where $U = \text{circ}(0, 1, 0, \dots, 0)$. Therefore, any two circulant matrices of order n with commuting entries commute. If $C = \text{circ}(a_1, a_2, \dots, a_n)$, then $C^* = \text{circ}(\bar{a}_1, \bar{a}_n, \dots, \bar{a}_2)$.

We use the notation $u_{(k)}$ to show u repeats k times. Suppose that A and B are two sequences of length n such that A is quasireverse to B . Let $D =$

$\text{circ}(0_{(a+1)}, A, 0_{(2b+1)}, B, 0_{(a)})$, where a and b are nonnegative integers and let $m = 2a + 2b + 2n + 2$. Then $D^* = \text{circ}(0_{(a+1)}, B_{\overline{R}}, 0_{(2b+1)}, A_{\overline{R}}, 0_{(a)})$ and $\text{abs}(D) = \text{abs}(D^*)$. Hence, D is a quasisymmetric circulant matrix of order m .

The *non-periodic autocorrelation function* [9] of a sequence $A = (x_1, \dots, x_n)$ of commuting square complex matrices of order m , is defined by

$$N_A(j) := \begin{cases} \sum_{i=1}^{n-j} x_{i+j} x_i^* & \text{if } j = 0, 1, 2, \dots, n-1 \\ 0 & j \geq n \end{cases}$$

where x_i^* denotes the conjugate transpose of x_i . A set $\{A_1, A_2, \dots, A_\ell\}$ of sequences (not necessarily in the same length) is said to have zero autocorrelation if for all $j > 0$, $\sum_{k=1}^{\ell} N_{A_k}(j) = 0$. Sequences having zero autocorrelation are called *complementary*.

A pair $(A; B)$ of $\{\pm 1\}$ -complementary sequences of length n is called a *Golay pair* of length n , and a pair $(A_1; B_1)$ of $\{\pm x, \pm y\}$ -complementary sequences of length n_1 is called a *Golay pair in two variables x and y* of length n_1 . The length n is called *Golay number*. Similarly, a pair $(C; D)$ of $\{\pm 1, \pm i\}$ -complementary sequences of length m is called a *complex Golay pair* of length m , and a pair $(C_1; D_1)$ of $\{\pm x, \pm ix, \pm y, \pm iy\}$ -complementary sequences of length m_1 is called a *complex Golay pair in two variables x and y* of length m_1 . The length m is called *complex Golay number*. In this paper, the sequences A_1 and C_1 are assumed to be quasireverse to B_1 and D_1 , respectively.

Craigen, Holzmann and Kharaghani in [3] showed that if g_1 and g_2 are complex Golay numbers and g is an even Golay number, then gg_1g_2 is a complex Golay number. Using this, they showed the following theorem.

Theorem 1. *All numbers of the form $m = 2^{a+u}3^b5^c11^d13^e$ are complex Golay numbers, where a, b, c, d, e and u are non-negative integers such that $b + c + d + e \leq a + 2u + 1$ and $u \leq c + e$.*

The following lemma is immediate from the definition of complex Golay pair.

Lemma 1. *Suppose that $(A; B)$ is a complex Golay pair of length m . Then $((xA, yB); (yA, -xB))$ is a complex Golay pair of length $2m$ in two variables x and y .*

From Theorem 1 and Lemma 1, we have the following result.

Corollary 1. *There is a complex Golay pair in two variables of length $n = 2^{a+u+1}3^b5^c11^d13^e$, where a, b, c, d, e and u are non-negative integers such that $b + c + d + e \leq a + 2u + 1$ and $u \leq c + e$.*

In Section 2, we introduce signed group orthogonal designs, and will show some of their properties. Then as one of their applications, in Theorem 4, we show how to obtain orthogonal designs from signed group orthogonal designs. In Section 4, using signed group orthogonal designs, we prove Theorems 9 and 10 that give two different bounds for the asymptotic existence of orthogonal designs, namely, for any k -tuple (u_1, u_2, \dots, u_k) of positive integers, there is an integer $N = N(u_1, u_2, \dots, u_k)$

such that a full orthogonal design of type $(2^n u_1, 2^n u_2, \dots, 2^n u_k)$ exists for each $n \geq N$.

In this paper, $P := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Q := \begin{bmatrix} 1 & 0 \\ 0 & - \end{bmatrix}$, $R := \begin{bmatrix} 0 & 1 \\ - & 0 \end{bmatrix}$ and I_d is the identity matrix of order d , where $-$ is -1 .

2 Signed group orthogonal designs and some of their properties

A *signed group orthogonal design*, SOD, of type (u_1, \dots, u_k) , where u_1, \dots, u_k are positive integers, and of order n , is a square matrix X of order n with entries from $\{0, \varepsilon_1 x_1, \dots, \varepsilon_k x_k\}$, where the x_i 's are variables and $\varepsilon_j \in S$, $1 \leq j \leq k$, for some signed group S , that satisfies

$$XX^* = \left(\sum_{i=1}^k u_i x_i^2 \right) I_n.$$

We denote it by $SOD(n; u_1, \dots, u_k)$.

Equating all variables to 1 in any SOD of order n results in a *signed group weighing matrix* of order n and weight w which is denoted by $SW(n, w)$, where w is the number of nonzero entries in each row (column) of the SOD. We call an SOD with no zero entries a full SOD. Equating all variables to 1 in any full SOD of order n results in a *signed group Hadamard matrix* of order n which is denoted by $SH(n, S)$.

Craigen [2] proved the following fundamental theorem and applied it to demonstrate a novel and new method for the asymptotic existence of signed group Hadamard matrices and consequently Hadamard matrices.

Theorem 2. *For any odd positive integer p , there exists a circulant $SH(2p, SP_{2^{N(p)}-1})$.*

Remark 1. An SOD over the Quaternion signed group S_Q is called a *Quaternion orthogonal design*, QOD. An SOD over the complex signed group $S_{\mathbb{C}}$ is called a *complex orthogonal design*, COD. An SOD over the trivial signed group $S_{\mathbb{R}}$ is called an *orthogonal design*, OD.

Lemma 2. *Every $SW(n, w)$ over a finite signed group is normal.*

Proof. Suppose that $WW^* = wI_n$, where the entries in W belong to a signed group S of order m . We show that $WW^* = W^*W$. The space of all square matrices of order n with entries in $\mathbb{R}[S]$ has the standard basis with mn^2 elements over the field \mathbb{R} . Thus, there exists an integer u such that

$$c_1 W + c_2 W^2 + \dots + c_u W^u = 0,$$

where $c_u \neq 0$, and $c_i \in \mathbb{R}$ ($1 \leq i \leq u$). Multiplying the above equality from the right by $(W^*)^{u-1}$,

$$c_1 w(W^*)^{u-2} + c_2 w^2(W^*)^{u-3} + \cdots + c_u w^{u-1} W = 0.$$

Hence W is a polynomial in W^* , and so $WW^* = W^*W$. \square

Theorem 3. *A necessary and sufficient condition that there is a $SOD(n; u_1, \dots, u_k)$ over a signed group S , is that there exists a family $\{A_1, \dots, A_k\}$ of pairwise disjoint square matrices of order n with entries from $\{0, S\}$ satisfying*

$$A_i A_i^* = u_i I_n, \quad 1 \leq i \leq k, \quad (1)$$

$$A_i A_j^* = -A_j A_i^*, \quad 1 \leq i \neq j \leq k. \quad (2)$$

Proof. Suppose that there is a $A = SOD(n; u_1, \dots, u_k)$ over a signed group S . One can write

$$A = \sum_{m=1}^k x_m A_m, \quad (3)$$

where the A_i 's are square matrices of order n with entries from $\{0, S\}$. Since the entries in A are linear monomials in the x_i , the A_i 's are disjoint. Since A is an SOD,

$$AA^* = \left(\sum_{i=1}^k u_i x_i^2 \right) I_n, \quad (4)$$

and so by using (3),

$$\sum_{m=1}^k x_m^2 A_m A_m^* + \sum_{i=1}^k \sum_{j=i+1}^k x_i x_j (A_i A_j^* + A_j A_i^*) = \left(\sum_{i=1}^k u_i x_i^2 \right) I_n. \quad (5)$$

In the above equality, for each $1 \leq i \leq k$, let $x_i = 1$ and $x_j = 0$ for all $1 \leq j \leq k$ and $j \neq i$, to get (1) and therefore (2).

On the other hand, if $\{A_1, \dots, A_k\}$ are pairwise disjoint square matrices of order n with entries from $\{0, S\}$ which satisfy (1) and (2), then the left hand side of the equality (5) gives us (4). \square

Remark 2. Equation (4) implies equations (1) and (2). Multiply (2) from the left by A_i^* and then from the right by A_i to get $A_j^* A_i = -A_i^* A_j$ for $1 \leq i \neq j \leq k$. Therefore, by Lemma 2,

$$A^* A = \sum_{m=1}^k x_m^2 A_m^* A_m + \sum_{i=1}^k \sum_{j=i+1}^k x_i x_j (A_i^* A_j + A_j^* A_i) = \left(\sum_{i=1}^k u_i x_i^2 \right) I_n.$$

Thus, $AA^* = A^*A$. It means that every SOD over a finite signed group is normal.

Lemma 3. *There does not exist any full SOD of order $n > 1$, if n is odd.*

Proof. Assume that there is a full SOD of order $n > 1$ over a signed group S . Equating all variables to 1 in the SOD, one obtains a $SH(n, S) = [h_{ij}]_{i,j=1}^n$. One may multiply each column of the $SH(n, S)$, from the right, by the inverse of corresponding

entry of its first row, \bar{h}_{1j} , to get an equivalent $SH(n, S)$ with the first row all 1 (see [2, 3] for the definition of equivalence). By orthogonality of the rows of the $SH(n, S)$, the number of occurrences of a given element $s \in S$ in each subsequent row must be equal to the number of occurrences of $-s$. Therefore, n has to be even. \square

3 Some applications of signed group orthogonal designs

In this section, we adapt the methods of Livinskyi [13] to obtain generalizations and improvements of his results about Hadamard matrices in the much more general setting of ODs.

Suppose that we have a remrep $\pi : S \rightarrow SP_m$. We extend this remrep to a ring homomorphism $\pi : \mathbb{R}[S] \rightarrow M_m[\mathbb{R}]$ linearly by $\pi(r_1 s_1 + \cdots + r_n s_n) = r_1 \pi(s_1) + \cdots + r_n \pi(s_n)$. Since for every matrix $A \in SP_m$ we have $A^{-1} = A^t$, for every $s \in S$, $\pi(\bar{s}) = \pi(s)^{-1} = \pi(s)^t$.

Next theorem shows how one can obtain ODs from SODs.

Theorem 4. *Suppose that there exists a $SOD(n; u_1, \dots, u_k)$ for some signed group S equipped with a remrep π of degree m , where m is the order of a Hadamard matrix. Then there is an $OD(mn; mu_1, \dots, mu_k)$.*

Proof. Suppose that there exists a $SOD(n; u_1, \dots, u_k)$ for some signed group S . By Theorem 3, there are pairwise disjoint matrices A_1, \dots, A_k of order n with entries in $\{0, S\}$ such that

$$A_\alpha A_\alpha^* = u_\alpha I_n, \quad 1 \leq \alpha \leq k, \quad (6)$$

$$A_\alpha A_\beta^* = -A_\beta A_\alpha^*, \quad 1 \leq \alpha \neq \beta \leq k. \quad (7)$$

Let $\pi : S \rightarrow SP_m$ be a remrep of degree m , and H be a Hadamard matrix of degree m . Also, for each $1 \leq \alpha \leq k$, let

$$B_\alpha = \left[\pi(A_\alpha[i, j]) H \right]_{i,j=1}^n.$$

By Proposition 1.1 in [6], it is sufficient to show that B_α 's are pairwise disjoint matrices of order mn , with $\{0, \pm 1\}$ entries such that

$$B_\alpha B_\alpha^t = mu_\alpha I_{mn}, \quad 1 \leq \alpha \leq k, \quad (8)$$

$$B_\alpha B_\beta^t = -B_\beta B_\alpha^t, \quad 1 \leq \alpha \neq \beta \leq k. \quad (9)$$

Since A_α 's are pairwise disjoint, so are B_α 's (see [6, chap. 1] for Hurwitz-Radon matrices and their properties). Let $1 \leq \alpha \neq \beta \leq k$ and $1 \leq i, j \leq n$. Then

$$\begin{aligned}
(B_\alpha B_\beta^t)[i, j] &= \sum_{k=1}^n \pi(A_\alpha[i, k]) H H^t \pi(A_\beta[j, k])^t \\
&= m \sum_{k=1}^n \pi(A_\alpha[i, k]) \pi(\bar{A}_\beta[j, k]) \\
&= m \pi \left(\sum_{k=1}^n A_\alpha[i, k] \bar{A}_\beta[j, k] \right) \\
&= m \pi \left((A_\alpha A_\beta^*)[i, j] \right) \tag{10}
\end{aligned}$$

$$\begin{aligned}
&= m \pi \left((-A_\beta A_\alpha^*)[i, j] \right) \quad \text{from (7)} \\
&= -m \pi \left((A_\beta A_\alpha^*)[i, j] \right) \tag{11}
\end{aligned}$$

On the other hand, similarly,

$$\begin{aligned}
(B_\beta B_\alpha^t)[i, j] &= \sum_{k=1}^n \pi(A_\beta[i, k]) H H^t \pi(A_\alpha[j, k])^t \\
&= m \sum_{k=1}^n \pi(A_\beta[i, k]) \pi(\bar{A}_\alpha[j, k]) \\
&= m \pi \left(\sum_{k=1}^n A_\beta[i, k] \bar{A}_\alpha[j, k] \right) \\
&= m \pi \left((A_\beta A_\alpha^*)[i, j] \right). \tag{12}
\end{aligned}$$

Comparing (11) and (12), one obtains (9). If $\alpha = \beta$ in (10), then for $1 \leq i, j \leq n$,

$$\begin{aligned}
(B_\alpha B_\alpha^t)[i, j] &= m \pi \left((A_\alpha A_\alpha^*)[i, j] \right) \\
&= m \pi (\gamma_{ij} u_\alpha \cdot 1_S) \quad \text{from (6)} \\
&= m \gamma_{ij} u_\alpha I_m,
\end{aligned}$$

where $\gamma_{ij} = 1$ if $i = j$, and 0 otherwise. Whence (8) follows. \square

In the following two corollaries, it is shown how to obtain ODs from CODs and QODs.

Corollary 2. *If there exists a COD($n; u_1, \dots, u_k$), then there exists an OD($2n; 2u_1, \dots, 2u_k$).*

Proof. A COD($n; u_1, \dots, u_k$) can be viewed as a SOD($n; u_1, \dots, u_k$) over the complex signed group $S_{\mathbb{C}}$. It can be seen that $\pi : S_{\mathbb{C}} \rightarrow SP_2$ defined by

$$i \longrightarrow R = \begin{bmatrix} 0 & 1 \\ - & 0 \end{bmatrix}$$

is a remrep of degree 2, and so by Theorem 4, there exists an $OD(2n; 2u_1, \dots, 2u_k)$. \square

Corollary 3. *If there exists a $QOD(n; u_1, \dots, u_k)$, then there exists an $OD(4n; 4u_1, \dots, 4u_k)$.*

Proof. A $QOD(n; u_1, \dots, u_k)$ can be viewed as a $SOD(n; u_1, \dots, u_k)$ over the Quaternion signed group S_Q . It can be seen that $\pi : S_Q \rightarrow SP_4$ defined by

$$j \longrightarrow R \otimes I_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ - & 0 & 0 & 0 \\ 0 & - & 0 & 0 \end{bmatrix} \quad \text{and} \quad k \longrightarrow P \otimes R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & - & 0 \\ 0 & 1 & 0 & 0 \\ - & 0 & 0 & 0 \end{bmatrix},$$

is a remrep of degree 4, and so by Theorem 4, there exists an $OD(4n; 4u_1, \dots, 4u_k)$. \square

Following similar techniques in [2, 4, 13], we have the following Lemma.

Lemma 4. *Suppose that A and B are two disjoint circulant matrices of order d with entries from $\{0, \varepsilon_1 x_1, \dots, \varepsilon_k x_k\}$, where the x_ℓ 's are variables, $\varepsilon_\ell \in S$ ($1 \leq \ell \leq k$) for A and $\varepsilon_\ell \in Z(S)$, the center of S , ($1 \leq \ell \leq k$) for B . Also, assume A is normal. If*

$$C = \begin{bmatrix} A+B & A-B \\ A^* - B^* & -A^* - B^* \end{bmatrix},$$

then $CC^ = C^*C = 2I_2 \otimes (AA^* + BB^*)$. Moreover, if A and B are both quasisymmetric and S has a faithful remrep of degree m , then there exists a circulant quasisymmetric normal matrix D of order d with entries from $\{0, \varepsilon'_1 x_1, \dots, \varepsilon'_k x_k\}$ and the same support as $A+B$ such that $DD^* = AA^* + BB^*$, where $\varepsilon'_\ell \in S'$ ($1 \leq \ell \leq k$), and $S' \geq S$ is a signed group having a faithful remrep of degree $2m$.*

Proof. It may be verified directly that $CC^* = C^*C = 2I_2 \otimes (AA^* + BB^*)$. To find matrix D , first reorder the rows and columns of C to get matrix D_0 which is a partitioned matrix of order $2d$ into 2×2 blocks whose entries are the (i, j) , $(i+d, j)$, $(i, j+d)$ and $(i+d, j+d)$ entries of C , $1 \leq i, j \leq d$. Applying the same reordering to $2I_2 \otimes (AA^* + BB^*)$, one obtains $(AA^* + BB^*) \otimes 2I_2$. Since A and B are disjoint and quasisymmetric, each non-zero block of D_0 will have one of the following forms

$$\begin{bmatrix} \varepsilon_i x_i & \varepsilon_i x_i \\ \varepsilon_j x_i & -\varepsilon_j x_i \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \varepsilon_i x_i & -\varepsilon_i x_i \\ \varepsilon_j x_i & \varepsilon_j x_i \end{bmatrix},$$

where $\varepsilon_\ell \in S$. Multiplying D_0 on the right by $\frac{1}{2}I_d \otimes \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix}$ yields a matrix D_1 of order $2d$ with entries from $\{0, \varepsilon_1 x_1, \dots, \varepsilon_k x_k\}$ whose non-zero 2×2 blocks have one of the forms $A_i x_i$ or $B_i x_i$, where

$$A_i = \begin{bmatrix} \varepsilon_i & 0 \\ 0 & \varepsilon_j \end{bmatrix} \quad \text{or} \quad B_i = \begin{bmatrix} 0 & \varepsilon_i \\ \varepsilon_j & 0 \end{bmatrix}, \quad (13)$$

and such that $D_1 D_1^* = D_1^* D_1 = (AA^* + BB^*) \otimes I_2$. The A_i 's and B_i 's in (13) form another signed group, S' . Now matrices of the form

$$\varepsilon \otimes I_2 = \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{bmatrix}, \quad \varepsilon \in S,$$

form a signed subgroup of S' which is isomorphic to S . Therefore, one can identify this signed subgroup with S itself and consider S' as an extension of S . Replacing every 2×2 block of D_1 which is one the forms in (13) or zero with corresponding $\varepsilon'_i x_i$, $\varepsilon'_i \in S'$ or zero, gives the required matrix D . Note that we identify $\varepsilon \otimes I_2 \in S'$ with $\varepsilon \in S$.

Now if $\pi : S \rightarrow SP'_m \leq SP_m$ is a faithful remrep of degree m , then it can be verified directly that the map $\pi' : S' \rightarrow SP'_{2m} \leq SP_{2m}$ which is uniquely defined by

$$\begin{bmatrix} \varepsilon_i & 0 \\ 0 & \varepsilon_j \end{bmatrix} \rightarrow \begin{bmatrix} \pi(\varepsilon_i) & 0_m \\ 0_m & \pi(\varepsilon_j) \end{bmatrix}, \quad \begin{bmatrix} 0 & \varepsilon_i \\ \varepsilon_j & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0_m & \pi(\varepsilon_i) \\ \pi(\varepsilon_j) & 0_m \end{bmatrix},$$

is a faithful remrep of degree $2m$, where 0_m denotes the zero matrix of order m .

Finally, since A and B are circulant, C consists of four circulant blocks, so D_0 and D_1 are block-circulant with block size 2×2 ; whence D is circulant and quasisymmetric. \square

We now use Lemma 4 and follow similar techniques in [2, 13] to show the following Theorem.

Theorem 5. *Suppose that B_1, \dots, B_n are disjoint quasisymmetric circulant matrices of order d with entries from $\{0, \varepsilon_1 x_1, \dots, \varepsilon_k x_k\}$, where $\varepsilon_\ell \in S_\mathbb{C}$, and the x_ℓ 's are variables ($1 \leq \ell \leq k$), such that*

$$B_1 B_1^* + \dots + B_n B_n^* = \left(\sum_{\ell=1}^k u_\ell x_\ell^2 \right) I_d,$$

where the u_ℓ 's are positive integers. Then there is a quasisymmetric circulant $SOD(d; u_1, \dots, u_k)$ for a signed group S that admits a faithful remrep of degree 2^n .

Proof. $S_\mathbb{C}$ has a faithful remrep $\pi : S_\mathbb{C} \rightarrow SP'_2 \leq SP_2$ of degree 2 uniquely determined by $\pi(i) = R$, where $SP'_2 = \langle R; R^2 = -I \rangle$. Applying Lemma 4 to matrices B_1 and B_2 , one obtains a quasisymmetric normal circulant matrix A_1 of order d with entries from $\{0, \varepsilon_1^{(1)} x_1, \dots, \varepsilon_k^{(1)} x_k\}$, where $\varepsilon_\ell^{(1)} \in S_1$ ($1 \leq \ell \leq k$) such that $S_1 \geq S_\mathbb{C}$ is a signed group with a faithful remrep of degree 2^2 . Also, $A_1 A_1^* = B_1 B_1^* + B_2 B_2^*$. Since $\text{supp}(A_1)$ is the union of $\text{supp}(B_1)$ and $\text{supp}(B_2)$, A_1 is disjoint from B_3, \dots, B_n .

Suppose that one has constructed a circulant quasisymmetric normal matrix A_r of order d with entries from $\{0, \varepsilon_1^{(r)} x_1, \dots, \varepsilon_k^{(r)} x_k\}$, where $\varepsilon_\ell^{(r)} \in S_r$ ($1 \leq \ell \leq k$) such that $S_r \geq S_{r-1}$ is a signed group with a faithful remrep $\pi_r : S_r \rightarrow SP'_{2^{r+1}} \leq SP_{2^{r+1}}$ of degree 2^{r+1} . Moreover, A_r is disjoint from B_{r+2}, \dots, B_n and

$$A_r A_r^* = B_1 B_1^* + \dots + B_{r+1} B_{r+1}^*.$$

By the assumption, B_{r+2} is a quasisymmetric normal circulant matrix with entries from $\{0, \varepsilon_1 x_1, \dots, \varepsilon_k x_k\}$, where $\varepsilon_\ell \in S_{\mathbb{C}}$ ($1 \leq \ell \leq k$). One can view the ε_ℓ 's as elements in $Z(S_r)$ because we identified these elements as blocks $\pm I_{2^r} \otimes R$ and $\pm I_{2^{r+1}}$ in the proof of Lemma 4 which commute with $\pi_r(\varepsilon_\ell^{(r)})$, $1 \leq \ell \leq k$. Therefore, by Lemma 4, there is a quasisymmetric normal circulant matrix A_{r+1} with entries from $\{0, \varepsilon_1^{(r+1)} x_1, \dots, \varepsilon_k^{(r+1)} x_k\}$, where $\varepsilon_\ell^{(r+1)} \in S_{r+1}$ ($1 \leq \ell \leq k$) such that $S_{r+1} \geq S_r$ is a signed group with a faithful remrep of degree 2^{r+2} . Also,

$$A_{r+1} A_{r+1}^* = A_r A_r^* + B_{r+2} B_{r+2}^* = B_1 B_1^* + \dots + B_{r+1} B_{r+1}^* + B_{r+2} B_{r+2}^*,$$

and by the same argument A_{r+1} is disjoint from B_{r+3}, \dots, B_n . Applying this procedure $n-2$ times, there is a quasisymmetric normal circulant matrix A_{n-1} of order d such that

$$A_{n-1} A_{n-1}^* = B_1 B_1^* + \dots + B_n B_n^* = \left(\sum_{\ell=1}^k u_\ell x_\ell^2 \right) I_d,$$

which is a circulant quasisymmetric $SOD(d; u_1, \dots, u_k)$ with the signed group $S = S_{n-1} \geq S_{n-2} \geq \dots \geq S_{\mathbb{C}}$ that admits a faithful remrep of degree 2^n . \square

Remark 3. The circulant matrices in Theorem 5 are taken on the abelian signed group $S_{\mathbb{C}}$; however, if the signed group is not abelian, the circulant matrices that obtain from Lemma 4 do not necessarily commute, and Theorem 5 may fail. As an example, if $B_1 = \text{circ}(j, 0)$ and $B_2 = \text{circ}(0, k)$, where $j, k \in S_Q$, then since $jk = -kj$, $B_1 B_2 \neq B_2 B_1$. Therefore, Lemma 4 does not apply in this case.

Theorem 6. Suppose that B_1, \dots, B_n are disjoint quasisymmetric circulant matrices of order d with entries from $\{0, \varepsilon_1 x_1, \dots, \varepsilon_k x_k\}$, where $\varepsilon_\ell \in S_{\mathbb{R}}$, and the x_ℓ 's are variables ($1 \leq \ell \leq k$), such that

$$B_1 B_1^* + \dots + B_n B_n^* = \left(\sum_{\ell=1}^k u_\ell x_\ell^2 \right) I_d,$$

where the u_ℓ 's are positive integers. Then there is a circulant quasisymmetric $SOD(d; u_1, \dots, u_k)$ for a signed group S that admits a faithful remrep of degree 2^{n-1} .

Proof. Similar to the proof of Theorem 5, but in here since $S_{\mathbb{R}}$ has the trivial remrep of degree 1, the final signed group S will have a remrep of degree 2^{n-1} . \square

Example 2. We explain how to use Theorem 6 to find a $SOD(12; 4, 4, 4)$ for a signed group S that admits a remrep of degree 8. Consider the following disjoint quasisymmetric circulant matrices of order 12:

$$\begin{aligned} B_1 &= \text{circ}(a, 0, 0, 0, 0, 0, a, 0, 0, 0, 0, 0), \\ B_2 &= \text{circ}(0, 0, 0, a, 0, 0, 0, 0, 0, 0, -a, 0), \\ B_3 &= \text{circ}(0, b, c, 0, 0, 0, 0, 0, 0, 0, c, -b), \\ B_4 &= \text{circ}(0, 0, 0, 0, c, -b, 0, -b, -c, 0, 0, 0). \end{aligned}$$

Thus, $B_1B_1^* + B_2B_2^* + B_3B_3^* + B_4B_4^* = (4a^2 + 4b^2 + 4c^2)I_{12}$. Apply Lemma 4 to B_1 and B_2 to get a quasisymmetric normal circulant matrix of order 12:

$$A_1 = \text{circ}(1a, 0, 0, \delta a, 0, 0, 1a, 0, 0, -\delta a, 0, 0),$$

where δ is in the signed group of order 2:

$$S_1 = \langle -1, \delta; \delta^2 = 1 \rangle$$

which admits a remrep of degree 2 uniquely determined by $1 \rightarrow I_2$ and $\delta \rightarrow P$. Since B_1 and B_2 are complementary, it follows that $A_1A_1^* = 4a^2I_{12}$.

Applying Lemma 4 again to A_1 and B_3 , there is a quasisymmetric normal circulant matrix of order 12:

$$A_1 = \text{circ}(1a, \gamma_1 b, \gamma_2 c, \gamma_3 a, 0, 0, 1a, 0, 0, -\gamma_3 a, \gamma_2 c, -\gamma_1 b),$$

where $\gamma_1, \gamma_2, \gamma_3$ belong to the signed group of order 2^3 :

$$S_2 = \langle \gamma_1, \gamma_2, \gamma_3; \gamma_1^2 = -\gamma_2^2 = \gamma_3^2 = 1, \alpha\beta = -\beta\alpha; \alpha, \beta \in \{\gamma_1, \gamma_2, \gamma_3\} \rangle,$$

with a remrep of degree 4 which is uniquely determined by

$$\gamma_1 \rightarrow P \otimes I_2, \quad \gamma_2 \rightarrow R \otimes I_2, \quad \gamma_3 \rightarrow Q \otimes P.$$

Note that A_2 is not an SOD because B_1, B_2 and B_3 are not complementary.

Finally, apply Lemma 4 to A_2 and B_4 to get a quasisymmetric normal circulant matrix of order 12:

$$A_3 = \text{circ}(1a, \varepsilon_1 b, \varepsilon_2 c, \varepsilon_3 a, \varepsilon_4 c, -\varepsilon_5 b, 1a, -\varepsilon_5 b, -\varepsilon_4 c, -\varepsilon_3 a, \varepsilon_2 c, -\varepsilon_1 b),$$

where $\varepsilon_j, 1 \leq j \leq 5$ belong to the signed group of order 2^5 :

$$S = \langle \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5; \varepsilon_1^2 = -\varepsilon_2^2 = \varepsilon_3^2 = \varepsilon_4^2 = -\varepsilon_5^2 = 1, \alpha\beta = -\beta\alpha; \alpha, \beta \in \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5\} \rangle,$$

with a remrep of degree 8 which is uniquely determined by

$$\varepsilon_1 \rightarrow Q \otimes P \otimes I_2, \quad \varepsilon_2 \rightarrow Q \otimes R \otimes I_2, \quad \varepsilon_3 \rightarrow Q \otimes Q \otimes P, \quad \varepsilon_4 \rightarrow P \otimes I_2 \otimes I_2, \quad \varepsilon_5 \rightarrow R \otimes I_2 \otimes I_2.$$

So A_3 is a quasisymmetric circulant $SOD(12; 4, 4, 4)$. By Theorem 4, there is an

$$OD(8 \cdot 12; 8 \cdot 4, 8 \cdot 4, 8 \cdot 4).$$

Although Theorem 6 shows that the degree of remrep is 2 times less than the one in Theorem 5, we have more complex Golay pairs than real ones. Thus, from now on, we just consider the complex case, and we refer the reader to [7, chap. 6] for the results that obtain from the real case.

For u a positive integer, denote by $\ell c(u)$ the least number of complex Golay numbers that add up to u , and let $\ell c(0) = 0$. Also, denote by $\ell' c(u)$ the least number of complex Golay numbers in two variables that add up to u . Indeed, $\ell' c(2u) \leq \ell c(u)$. Note that Lemma 1 insures the existence of a complex Golay pair in two variables of length $2m$ if there exists a complex Golay pair of length m .

In the following theorem, we show how to use complex Golay pair and complex Golay pairs in two variables to construct SODs.

Theorem 7. *Let $(1, v_1, \dots, v_q, w_1, w_1, \dots, w_t, w_t)$ be a sequence of positive integers such that v_i 's, $1 \leq i \leq q$, are disjoint and let $1 + \sum_{\beta=1}^q v_\beta + 2 \sum_{\delta=1}^t w_\delta = u$. Then there exists a full circulant quasisymmetric SOD $(4u; 4, 4v_1, \dots, 4v_q, 4w_1, 4w_1, \dots, 4w_t, 4w_t)$ for some signed group S that admits a remrep of degree 2^n , where $n \leq 2 + 2 \sum_{\beta=1}^q \ell c(v_\beta) + 2 \sum_{\delta=1}^t \ell' c(2w_\delta)$.*

Proof. For each β , $1 \leq \beta \leq q$, and each α , $1 \leq \alpha \leq \ell c(v_\beta)$, let $(A[\alpha, v_\beta]; B[\alpha, v_\beta])$ be a complex Golay pair in one variable x_β of length $V[\alpha, v_\beta]$. From the definition of $\ell c(v_\beta)$, for each β , $1 \leq \beta \leq q$, $\sum_{\alpha=1}^{\ell c(v_\beta)} V[\alpha, v_\beta] = v_\beta$. Let $S[\alpha, \beta] := \sum_{i=1}^{\alpha-1} V[i, v_\beta] + \sum_{j=1}^{\beta-1} v_j$. Also, for each δ , $1 \leq \delta \leq t$, and each γ , $1 \leq \gamma \leq \ell' c(2w_\delta)$, let $(C[\gamma, w_\delta]; D[\gamma, w_\delta])$ be a complex Golay pair of length $W[\gamma, w_\delta]$ in two variables y_δ and z_δ . From the definition of $\ell' c(2w_\delta)$, for each δ , $1 \leq \delta \leq t$, $\sum_{\gamma=1}^{\ell' c(2w_\delta)} W[\gamma, w_\delta] = 2w_\delta$. Let $S'[\gamma, \delta] := \sum_{i=1}^{\gamma-1} W[i, w_\delta] + 2 \sum_{j=1}^{\delta-1} w_j$. For each β , $1 \leq \beta \leq q$, and each α , $1 \leq \alpha \leq \ell c(v_\beta)$, and for each δ , $1 \leq \delta \leq t$ and each γ , $1 \leq \gamma \leq \ell' c(2w_\delta)$, the following are $n = 2 + 2 \sum_{\beta=1}^q \ell c(v_\beta) + 2 \sum_{\delta=1}^t \ell' c(2w_\delta)$ circulant matrices of order $4u$:

$$\begin{aligned} M_1 &= \text{circ}(x, 0_{(2u-1)}, x, 0_{(2u-1)}), \\ M_2 &= \text{circ}(0_{(u)}, -x, 0_{(2u-1)}, x, 0_{(u-1)}), \\ X_{\alpha\beta} &= \text{circ}(0_{(S[\alpha,\beta]+1)}, A[\alpha, v_\beta], 0_{(4u-2S[\alpha+1,\beta]-1)}, B[\alpha, v_\beta], 0_{(S[\alpha,\beta])}), \\ Y_{\alpha\beta} &= \text{circ}(0_{(2u-S[\alpha+1,\beta])}, -B[\alpha, v_\beta], 0_{(2S[\alpha,\beta]+1)}, A[\alpha, v_\beta], 0_{(2u-S[\alpha+1,\beta]-1)}), \\ Z_{\gamma\delta} &= \text{circ}(0_{(v+S'[\gamma,\delta]+1)}, C[\gamma, w_\delta], 0_{(4u-2v-2S'[\gamma+1,\delta]-1)}, D[\gamma, w_\delta], 0_{(v+S'[\gamma,\delta])}), \\ T_{\gamma\delta} &= \text{circ}(0_{(2u-v-S'[\gamma+1,\delta])}, -D[\gamma, w_\delta], 0_{(2S'[\gamma,\delta]+2v+1)}, C[\gamma, w_\delta], 0_{(2u-v-S'[\gamma+1,\delta]-1)}). \end{aligned}$$

It can be seen that the above circulant matrices are disjoint and quasisymmetric such that

$$\begin{aligned} \sum_{i=1}^2 M_i M_i^* + \sum_{\beta=1}^q \sum_{\alpha=1}^{\ell c(v_\beta)} (X_{\alpha\beta} X_{\alpha\beta}^* + Y_{\alpha\beta} Y_{\alpha\beta}^*) + \sum_{\delta=1}^t \sum_{\gamma=1}^{\ell' c(2w_\delta)} (Z_{\gamma\delta} Z_{\gamma\delta}^* + T_{\gamma\delta} T_{\gamma\delta}^*) \\ = 4 \left(x^2 + \sum_{\beta=1}^q (v_\beta x_\beta^2) + \sum_{\delta=1}^t (w_\delta y_\delta^2 + w_\delta z_\delta^2) \right) I_{4u}. \end{aligned}$$

Thus, by Theorem 5, there exists a full circulant quasisymmetric

$$SOD(4u; 4, 4v_1, \dots, 4v_q, 4w_1, 4w_1, \dots, 4w_t, 4w_t)$$

for a signed group S which admits a remrep of degree 2^n , where

$$n = 2 + 2 \sum_{\beta=1}^q \ell c(v_\beta) + 2 \sum_{\delta=1}^t \ell' c(2w_\delta) \leq 2 + 2 \sum_{\beta=1}^q \ell c(v_\beta) + 2 \sum_{\delta=1}^t \ell c(w_\delta).$$

□

Example 3. Consider the 4-tuple $(1, v_1, v_2, v_3) = (1, 5, 7, 17)$. By Theorem 7, there is a circulant quasisymmetric $SOD(4 \cdot 30; 4 \cdot 1, 4 \cdot 5, 4 \cdot 7, 4 \cdot 17)$, which admits a remrep of degree 2^n , where $n = 2 + 2\ell c(5) + 2\ell c(7) + 2\ell c(17) = 2 + 2 + 4 + 4 = 12$. By Theorem 4, there is an

$$OD(2^{14} \cdot 30; 2^{14} \cdot 1, 2^{14} \cdot 5, 2^{14} \cdot 7, 2^{14} \cdot 17).$$

Example 4. Let $(1, w_1, w_1, w_2, w_2, w_3, w_3, w_4, w_4) = (1, 3, 3, 5, 5, 11, 11, 13, 13)$. By Theorem 7, there is a circulant quasisymmetric

$$SOD(4 \cdot 65; 4 \cdot 1, 4 \cdot 3_{(2)}, 4 \cdot 5_{(2)}, 4 \cdot 11_{(2)}, 4 \cdot 13_{(2)}),$$

which admits a remrep of degree 2^n , where $n = 2 + 2\ell c(3) + 2\ell c(5) + 2\ell c(11) + 2\ell c(13) = 10$. By Theorem 4, there is an

$$OD(2^{12} \cdot 65; 2^{12} \cdot 1, 2^{12} \cdot 3_{(2)}, 2^{12} \cdot 5_{(2)}, 2^{12} \cdot 11_{(2)}, 2^{12} \cdot 13_{(2)}).$$

4 Bounds for the asymptotic existence orthogonal designs

In this section, we obtain some upper bounds for the degree of remrep in Theorem 7, and then we find some upper bounds for the asymptotic existence of ODs.

To get a better upper bound for the degree of remrep for any k -tuple (u_1, u_2, \dots, u_k) of positive integers, from now on, we assume that $\ell c(u_1) - \ell c(u_1 - 1)$ is greater than or equal to $\ell c(u_i) - \ell c(u_i - 1)$ for all $2 \leq i \leq k$. We also define $\log(0) = 0$, and in here the base of log is 2.

Livinskyi [13, chap. 5], by a computer search, showed that each positive integer u can be presented as sum of at most $3 \lfloor \log_{26}(u) \rfloor + 4$ complex Golay numbers. Thus

$$\ell c(u) \leq 3 \left\lfloor \frac{1}{26} \log(u) \right\rfloor + 4 \leq \frac{3}{26} \log(u) + 4. \quad (14)$$

Theorem 8. Suppose that (u_1, u_2, \dots, u_k) is a k -tuple of positive integers and let $u_1 + \dots + u_k = u$. Then there is a full circulant quasisymmetric $SOD(4u; 4u_1, 4u_2, \dots, 4u_k)$ for some signed group S that admits a remrep of degree 2^n , where $n \leq (3/13) \log(u_1 - 1) + (3/13) \sum_{i=2}^k \log(u_i) + 8k + 2$.

Proof. Apply Theorem 7 to the $(k+1)$ -tuple $(1, u_1 - 1, u_2, \dots, u_k)$. So there is a full circulant quasisymmetric $SOD(4u; 4u_1, 4u_2, \dots, 4u_k)$ for some signed group S that admits a remrep of degree 2^n , where $n \leq 2 + 2\ell c(u_1 - 1) + 2\sum_{i=2}^k \ell c(u_i)$. Use (14) to obtain the desired. \square

Remark 4. For any given k -tuple (u_1, u_2, \dots, u_k) of positive integers, one may write it as the $(k+1)$ -tuple $(1, u_1 - 1, u_2, \dots, u_k)$, and then sort its elements to get the $(k+1)$ -tuple $(1, v_1, \dots, v_q, w_1, w_1, \dots, w_t, w_t)$, where v_i 's are disjoint and then use Theorem 7 and (14) to obtain the following bound:

$$\begin{aligned} n &\leq 2 + 2 \sum_{i=1}^q \ell c(v_i) + 2 \sum_{j=1}^t \ell c(w_j) \\ &\leq 2 + \frac{3}{13} \sum_{i=1}^q \log(v_i) + \frac{3}{13} \sum_{j=1}^t \log(w_j) + 8(q+t), \end{aligned}$$

where n is the exponent of the degree of remrep.

By Theorem 4 and Theorem 8, we have the following asymptotic existence result.

Theorem 9. Suppose (u_1, u_2, \dots, u_k) is a k -tuple of positive integers. Then for each $n \geq N$, there is an $OD(2^n \sum_{j=1}^k u_j; 2^n u_1, \dots, 2^n u_k)$, where $N \leq (3/13) \log(u_1 - 1) + (3/13) \sum_{i=2}^k \log(u_i) + 8k + 4$.

Livinskyi [13, chap. 5] used complex Golay, Base, Normal and other sequences (see [5, 10, 11, 12]) to show that each positive integer u can be presented as sum of

$$s \leq \frac{1}{10} \log(u) + 5 \quad (15)$$

pairs $(A_k[u]; B_k[u])$ for $1 \leq k \leq s$ such that $A_k[u]$ and $B_k[u]$ have the same length for each k , $1 \leq k \leq s$, with elements from $\{\pm 1, \pm i\}$, and the set $\{A_1[u], B_1[u], \dots, A_s[u], B_s[u]\}$ is a set of complex complementary sequences with weight $2u$. In the following theorem, we use this set of complex complementary sequences.

Theorem 10. Suppose (v_1, v_2, \dots, v_k) is a k -tuple of positive integers. Then for each $n \geq N$, there is an $OD(2^n \sum_{j=1}^k v_j; 2^n v_1, \dots, 2^n v_k)$, where $N \leq (1/5) \log(v_1 - 1) + (1/5) \sum_{i=2}^k \log(v_i) + 10k + 4$.

Proof. Suppose (v_1, v_2, \dots, v_k) is a k -tuple of positive integer. Let $\sum_{j=1}^k v_j = v$. For simplicity, we assume that $u_1 = v_1 - 1$ and $u_i = v_i$ for $2 \leq i \leq k$.

For each β , $1 \leq \beta \leq k$, let $\{A_1[u_\beta], B_1[u_\beta], \dots, A_{s_\beta}[u_\beta], B_{s_\beta}[u_\beta]\}$ be a set of complex complementary sequences with weight $2u_\beta$ such that for each α , $1 \leq \alpha \leq s_\beta$, $A_\alpha[u_\beta]$ and $B_\alpha[u_\beta]$ have the same length, $V[\alpha, u_\beta]$. From (15), for each β , $1 \leq \beta \leq k$,

$$s_\beta \leq \frac{1}{10} \log u_\beta + 5. \quad (16)$$

Suppose that x and x_β , $1 \leq \beta \leq k$ are variables. Let $M_1 = \text{circ}(x, 0_{(2v-1)}, x, 0_{(2v-1)})$ and $M_2 = \text{circ}(0_{(v)}, -x, 0_{(2v-1)}, x, 0_{(v-1)})$. For each β , $1 \leq \beta \leq k$, and each α , $1 \leq \alpha \leq s_\beta$, let

$$X_{\alpha\beta} = \text{circ}\left(0_{(S[\alpha,\beta]+1)}, x_\beta A_\alpha[u_\beta], 0_{(4v-2S[\alpha+1,\beta]-1)}, x_\beta B_\alpha[u_\beta], 0_{(S[\alpha,\beta])}\right),$$

$$Y_{\alpha\beta} = \text{circ}\left(0_{(2v-S[\alpha+1,\beta])}, -x_\beta B_\alpha[u_\beta], 0_{(2S[\alpha,\beta]+1)}, x_\beta A_\alpha[u_\beta], 0_{(2v-S[\alpha+1,\beta]-1)}\right),$$

where $S[1, 1] = 0$ and $S[a, b] = \sum_{j=1}^{a-1} \sum_{i=1}^b V[j, u_i]$, for $1 \leq b \leq k$ and $1 < a \leq s_b + 1$. It can be seen that the above circulant matrices are disjoint and quasisymmetric of order $4v$ such that

$$\sum_{i=1}^2 M_i M_i^* + \sum_{\beta=1}^k \sum_{\alpha=1}^{s_\beta} (X_{\alpha\beta} X_{\alpha\beta}^* + Y_{\alpha\beta} Y_{\alpha\beta}^*) = 4 \left(x^2 + \sum_{\beta=1}^k (u_\beta x_\beta^2) \right) I_{4v}.$$

Thus, by Theorem 5, there exists a full circulant quasisymmetric $SOD(4v; 4, 4u_1, \dots, 4u_k)$ for a signed group S which admits a remrep of degree 2^m , where $m = 2 + 2 \sum_{\beta=1}^k s_\beta$. From Theorem 4 and the upper bounds for the s_β 's, (16), there is an $OD(2^n v; 2^n, 2^n u_1, \dots, 2^n u_k)$, and so there is an $OD(2^n v; 2^n v_1, \dots, 2^n v_k)$, where $n \leq (1/5) \log(v_1 - 1) + (1/5) \sum_{i=2}^k \log(v_i) + 10k + 4$. \square

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